Basics of Linear Optimization

- Examples
- Duality
- Polyhedrons
- Simplex-Algorithm

Example: Oil refinement

- Cracking raw oil to Light, Middle or Heavy oil
- There are two procedures:
- 1) 1 unit raw oil to 1 L, 2 M, 2 H
- 2) 1 unit raw oil to 4 L, 2 M, 1 H
- Costs: 1) 3 money units, 2) 5 money units
- Delivery commitments: 4 L, 5 M, 3 H
- Optimization:
  Minimize the total costs while satisfying all delivery commitments

Example: Oil refinement (Modellin)

- Introduce variables x_1 and x_2
  They represent units of raw oil cracked with procedure 1) resp. 2)
- Objective: minimize costs $3x_1 + 5x_2$
- Constraint: positiveness $x_1, x_2 \geq 0$
- Constraint: deliver at least 4 L $x_1 + 4x_2 \geq 4$
- Constraint: deliver at least 5 M $2x_1 + 2x_2 \geq 5$
- Constraint: deliver at least 3 H $2x_1 + x_2 \geq 3$

$$\begin{array}{rrcl} \min & 3x_1 + 5x_2 & & \\ \text{s.t.} & x_1 + 4x_2 & \geq & 4 \\ & 2x_1 + 2x_2 & \geq & 5 \\ & 2x_1 + x_2 & \geq & 3 \\ & x_1, x_2 & \geq & 0 \end{array}$$

Example: In the Marketplace

- We want to buy suitable amounts of potatoes, spinach and poultry

per 100g	Potatoes	Spinach	Poultry
Cost / cents	10	15	40
Protein / g	2	3	20
Carbohydrate /g	18	3	0
Calcium / mg	7	83	8
Iron / mg	0.6	2	1.4
Vitamin A / I.U.	0	7300	80

- Daily minimum requirements: 65g of protein, 90g of carbohydrate, 200mg of calcium, 10mg of iron, and 5000 I.U. of Vitamin A
- Optimization:
  Spend as less money as needed to satisfy all the requirements

Example: In the Marketplace (LP)

Variables x_1, x_2 and x_3 give the amount of potatoes, spinach and poultry

$$\begin{array}{llllll} \min & 40x_1 & +15x_2 & +10x_3 & & \\ \text{s.t.} & 20x_1 & +3x_2 & +2x_3 & \geq & 65 \\ & & 3x_2 & +18x_3 & \geq & 90 \\ & 8x_1 & +83x_2 & +7x_3 & \geq & 200 \\ & 1.4x_1 & +2x_2 & +0.6x_3 & \geq & 10 \\ & 80x_1 & +7300x_2 & & \geq & 5000 \\ & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

Example: In the Marketplace (Xpress I)

```
model Marketplace
 uses "mmxprs"

 declarations
  NProd   = 3
  NIncred = 5
  IP   =  1..NProd
  II   =  1..NIncred
  TAB:    array(II,IP) of real
  REQ:    array(II) of real
  PRICE:  array(IP) of real
  x:      array(IP) of mpvar
 end-declarations
```


Example: In the Marketplace (Xpress II)

```
TAB  := [20,      3,  2,
          0,       3, 18,
          8,      83,  7,
          1.4,     2,  0.6,
         80,    7300,  0]
 REQ  := [65, 90, 200, 10, 5000]
 PRICE:= [40, 15, 10]

 MinPrice := sum(p in IP) PRICE(p) * x(p)

 forall(i in II)
  sum(p in IP) TAB(i,p) * x(p) >= REQ(i)

 minimize(MinPrice)
```


Example: In the Marketplace (Xpress III)

```
 writeln("Objective: ", getobjval)
 forall(p in IP)
  write("Product",p,":",getsol(x(p)),"  ")
 writeln

end-model
```

Example: In the Marketplace (Duality)

- Assume we want to sell pills of protein, iron, vitamin A, etc.
- y_1 cents/gram of protein
- y_2 cents/gram of carbohydrate
- y_3 cents/mg of calcium
- y_4 cents/mg of iron
- y_5 cents/I.U. of vitamin A
- What are suitable prices for the pills?
- The costs of the incredients of 100g poultry shoult be cheaper than buying 100g poultry itself. Analogously for potatoes and spinach.
- We want to maximize our income

Example: In the Marketplace (dual LP)

$$
\begin{array}{lrrrrrcl}
\max & 65y_1 & +90y_2 & +200y_3 & +10y_4 & +5000y5 & & \\
\text{s.t.} & 20y_1 & & +8y_3 & +1.4y_4 & +80y_5 & \leq & 40 \\
 & 3y_1 & +3y_2 & +83y_3 & +2y_4 & +7300y_5 & \leq & 15 \\
 & 2y_1 & +18y_2 & +7y_3 & +0.6y_4 & & \leq & 10 \\
 & & & & & y_1, y_2, y_3, y_4, y_5 & \geq & 0
\end{array}
$$

Weak Duality Theorem

Primal linear program (P)

$\min \{c^T x \mid Ax \geq b, x \geq 0\}$

Dual linear program (D)

$\max \{b^T y \mid A^T y \leq c, y \geq 0\}$

Let $x_0 \in \{x \mid Ax \geq b, x \geq 0\}$ and $y_0 \in \{y \mid A^T y \leq c, y \geq 0\}$.

Then $b^T y_0 \leq c^T x_0$ holds.

Buildin the Dual LP

- equation $\longrightarrow$ free variable
- inequality $\longrightarrow$ signed variable
- signed variable $\longrightarrow$ inequality
- free variable $\longrightarrow$ equation
- objective function $\longrightarrow$ right hand side
- right hand side $\longrightarrow$ objective function

Farkas Lemma

Theorem (Farkas Lemma):
Either there are x, y fulfilling

$$\begin{array}{rrcl} Ax & +By & \leq & a \\ Cx & +Dy & = & b \\ & x & \geq & 0 \end{array}$$

or there are u, v fulfilling

$$\begin{array}{rrcl} u^T A & +v^T C & \geq & 0 \\ u^T B & +v^T D & = & 0 \\ & u & \geq & 0 \\ u^T a & +v^T b & < & 0. \end{array}$$

Duality Theorem

Let (P) and (D) be a primal-dual pair of LPs with (P) being a maximization and (D) a minimization problem. Let P and D be the sets of valid solutions of (P) and (D) and z^*, u^* the optimal solutions of (P) and (D). (z^* is $-\infty$ if $P = \emptyset$ and $+\infty$ if (P) is unbounded, u^* analog).
Then one of the following cases holds:

- $-\infty < z^* = u^* < +\infty \iff z^*$ finite $\iff u^*$ finite
- $z^* = +\infty \Rightarrow D = \emptyset$
- $u^* = -\infty \Rightarrow P = \emptyset$
- $P = \emptyset \Rightarrow D = \emptyset$ or $u^* = -\infty$
- $D = \emptyset \Rightarrow P = \emptyset$ or $z^* = +\infty$

Theorems of Complementary Slackness

Let (P) and (D) be a the following primal-dual pair of LPs:

$$(P)\ \max\{c^T x \mid Ax \leq b\} \qquad (D)\ \min\{u^T b \mid u^T A = c^T, u \geq 0\}.$$

Theorem of weak complementary slackness: Let $\overline{x}$ and $\overline{u}$ be feasible solutions of (P) and (D). Then they are optimal if and only if:

$$\overline{u}_i > 0 \Longrightarrow A_{i.}\overline{x} = b_i \qquad \forall i.$$

Theorem of strong complementary slackness: If there exist feasible solutions for both (P) and (D) then there exist optimal solutions $\overline{x}$ and $\overline{u}$ with:

$$\overline{u}_i > 0 \Longleftrightarrow A_{i.}\overline{x} = b_i \qquad \forall i.$$

Standard Formulations of an LP

$$\begin{array}{lrrcl} \max & c^T x & +d^T y & & \\ \text{s.t.} & Ax & +By & \geq & a \\ & Cx & +Dy & = & b \\ & & x & \geq & 0 \end{array}$$

$$\begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & \leq & b \end{array} \qquad \begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

Transformatin LP-Formulations

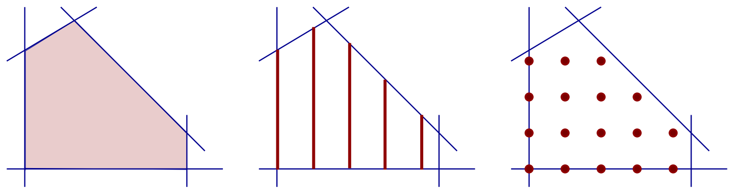
- signed variables $\longrightarrow$ free variables:
  $x_i \geq 0$ can be added to the system $Ax \leq b$.
- free variables $\longrightarrow$ signed variables:
  set $y_i = x_i^+ - x_i^-$ with $x_i^+, x_i^- \geq 0$.
- equations $\longrightarrow$ inequalities:
  replace $Ax = b$ by $Ax \leq b$ and $Ax \geq b$.
- inequalities $\longrightarrow$ equations:
  replace $Ax \leq b$ by $Ax + Iy = b$ and $y \geq 0$.

Optimization Problems

Linear program
$\max \{d^T y \mid By = a, Dy \leq b, y \geq 0\}$

Mixed-integer program
$\max \{c^T x + d^T y \mid Ax + By = a, Cx + Dy \leq b, y \geq 0, x \geq 0,\ x \text{ integer}\}$

Integer program
$\max \{c^T x \mid Ax = a, Cx \leq b, x \geq 0,\ x \text{ integer}\}$



Polyhedrons

- Definition: The set $P^=(A,b) = \{x \in R^n \mid Ax = b, x \geq 0\}$ is called polyhedron. If the set is bounded we call it polytope.
- Polyhedrons are convex, i.e.
  $x, y \in P^=(A,b) \Longrightarrow \lambda x + (1-\lambda)y \in P^=(A,b),\ \forall\, 0 \leq \lambda \leq 1.$
- $x \in P^=(A,b)$ is called vertex if it cannot be build as a proper convex combination of $y, z \in P^=(A,b)$.

Basic Theorems on Polyhedrons

- Let $P^=(A,b) \neq \emptyset$. $P^=(A,b)$ is a polytope iff $\nexists d \geq 0$ with $Ad = 0$.
- $x \in P^=(A,b)$ is vertex iff the columns of A corresponding to the positive entries of x are linearly independent. The number of vertices is finite and if $P^=(A,b) \neq \emptyset$ there is at least one vertex.
- Let $P^=(A,b) \neq \emptyset$ and V the set of verticies. Then any $x \in P^=(A,b)$ can be written as
  $$x = \sum_{v_i \in V} \lambda_i v_i + d$$
  with $\lambda_i \geq 0$, $\sum \lambda_i = 1$, $d \geq 0$ and $Ad = 0$.
- Given the program (P): $\max\{c^T x \mid Ax = b, x \geq 0\}$ with $P^=(A,b) \neq \emptyset$. Then either (P) is unbounded or one of the vertices of $P^=(A,b)$ is an optimal solution of (P).

Definition of the basis

Let $A \in R^{m \times n}$, $b \in R^m$ and $B \subset \{1, \dots, n\}$ defines a subset of the columns of A with $|B| = m$ and $A_{.i}$, $i \in B$, linearly independent. A_B denotes the corresponding submatrix of A and A_N the remainder.

- A_B is called basis and A_N nonbasis of A.
- $x = (x_B, x_N)$ with $x_B = A_B^{-1} b$ and $x_N = 0$ is called basic solution of the basis A_B.
- Let A_B be a basis. Then x_j, $j \in B$ are called basic variables and x_j, $j \in N$ are called nonbasic variables
- A basis A_B and the corresponding basic solution x are called feasible if $A_B^{-1} b \geq 0$ holds.
- A basic solution is called nondegenerate if $A_B^{-1} b > 0$ holds. Otherwise it is called degenerate.

Vertices and basic solutions

- Theorem.
  Let $P^{=}(A, b)$ be a polyhedron with $\text{rank}(A) = m < n$ and $x \in P$.
  x is vertex of $P^{=}(A, b)$ if and only if x is a basic feasible solution.
- We could simply calculate all basic solutions and evaluate them - but there are exponentially many: $\binom{n}{m}$

Simplex al orithm - main idea

- Idea of the Simplex-Method: start from one vertex and jump to a neighbour vertex with a better objective value until we reach the optimum
- How can we go from one vertex to another?
  Just replace one index in B!
- Two important things: choose a series of basic feasible solutions and increase (more exactly: do not decrease) objective value in each step

Revised Simplex

Input: Problemdata: A, b, c and feasible solution: B, A_B^{-1}, $\overline{b}$
Output: Solution of $\max\{c^T x \mid Ax = b, x \geq 0\}$

(1) BTRAN (Calculate shadow prices) $\pi^T := c_B^T A_B^{-1}$

(2) PRICE (Price out)
Compute the reduced costs coefficients

$$\overline{c}_j := (c_N^T)_j - \pi^T A_N e_j \text{ for } j = 1, \dots, n-m$$

and choose an index s with $\overline{c}_s > 0$ (otherwise stop: optimal)

(3) FTRAN (Generate pivot-column) $\overline{d} := A_B^{-1} A_{.s}$

(4) CHUZR (Ratio Test) $\lambda_0 := \min\{\frac{\overline{b}_i}{\overline{d}_i} \mid \overline{d}_i > 0, i = 1, \dots, m\}$
Choose index r with $\overline{d}_i > 0$ and $\frac{\overline{b}_i}{\overline{d}_i} = \lambda_0$ (otherwise stop: unbounded)

(5) WRETA (Update) Update the basis B, A_B^{-1}, $\overline{b}$ and goto (1)

Summary of the primal simplex

- Optimal solution always on a vertex corresponding to a basic feasible solution
- Two sets B, N of indices, variables in N fixed
- Exchanging two indices in each step which corresponds to moving to a neighbour vertex
- Calculate the shadow prices π and compare with objective vector c to see, if and in which direction the objective function gets better
- Always feasible and work towards optimality

Open questions

- Prove that algorithm terminates (problem: degeneracy $\Rightarrow$ cycling)
- How to get a feasible basis (phase I)?
- Which index i with $p_i > c_i$ to choose? $\Rightarrow$ pricing-strategies
- How can we efficiently treat bounds, slack variables, sparsity, matrix decompositions, updates?
- What about stability? How to avoid basis matrices with a bad condition (close to singularity)?

Findin a feasible start basis

- Two phases. In phase I we solve the problem $\min\{\sum_i s_i \mid Ax + s = b,\ x, s \geq 0\}$ starting with the feasible basis $s = b, x = 0$.
- If optimal solution has solution $s \neq 0$ the original problem is infeasible, else x is feasible for it (goto phase II).
- Problem: needs many iterations, whole basis must be exchanged at least once

Basics of the dual simplex

- Definition. A basis A_B of A is called primal feasible if $A_B^{-1}b \geq 0$, and dual feasible if the reduced costs $\overline{c} = c_N^T - c_B^T A_B^{-1} A_N \leq 0$.
- The corresponding basic solution x ($x_B = A_B^{-1}b$ and $x_N = 0$) is called primal feasible, and the basic solution $u^T = c_B^T A_B^{-1}$ is called dual feasible.
- Theorem. Let $P = \{u \mid u^T A \geq c^T\}$. The vector u is a vertex of P if and only if u is a dual feasible basic solution.
- Corollary. A basis A_B is optimal if and only if it is both primal feasible and dual feasible.

Dual Simplex

Input: Problemdata: A, b, c and dual feasible basis: B, A_B
Output: Solution of $\max\{c^T x \mid Ax = b, x \geq 0\}$

(1) If $\overline{b} = A_B^{-1} b \geq 0$ stop (current solution optimal)

(2) Choose an index r satisfying $\overline{b}_r < 0$.

(3) (Generate pivot-row) $w_N^T = e_r^T A_B^{-1} A_N = \overline{A}_{r.}$

(4) If $w_N \geq 0$ stop (dual problem unbounded)

(5) Compute $\lambda_0 := \min\{\frac{\overline{c}_j}{w_j} \mid w_j < 0, j = 1, \ldots, n-m\}$ and choose an index s with $\lambda_0 = \frac{\overline{c}_s}{w_s}$

(6) Compute $d = A_B^{-1} A_{.q_s}$

(7) Update the basis and goto (1)

Summary of the dual simplex

- Applying the dual algorithm to (P) is the same as applying the primal algorithm to (D)
- The dual of the dual is the primal again
- Feasible solutions of (P) and (D) bound one another
- Primal: first choose entering index, then decide which index has to leave the basis
- Dual: first choose leaving index, then decide which index has to enter the basis

Differences primal - dual simplex

- Dimensions of variables different: m and n
- Can solve the problem with either one, can have completely different behaviour (# of iterations)
- Adding a variable in (P): keep feasibility.
- Adding a variable in (D): loose feasibility!
- Adding a constraint in (D): keep feasibility.
- Adding a constraint in (P): loose feasibility!